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Foliated Schwarz symmetry for the nodal solution at the second minimax level

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Abstract We study the second minimax level λ_2 of the eigenvalue problem for the scalar field equation in $\mathbb{R}^N$. By using the tool of polarization, we prove that every nodal solution at the second minimax level is foliated Schwarz symmetric. As a consequence, we prove an **open problem** of Perera and Tintarev (Annali di Matematica Pura ed Applicata (4) 194(1):131–144, 2015).

Keywords Second minimax level · Nodal solution · Foliated Schwarz symmetry · Maximum principle

Mathematics Subject Classification 35J20 · 35J61 · 35P30

1 Introduction

In this paper, we consider variational problems of the form

$$\inf_{\gamma\in\Gamma_2}\max_{u\in\gamma(S^1)} J(u),$$

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where $\Gamma_2 := \{\gamma \in C(S^1, \mathcal{M}) : \gamma(-\theta) = -\gamma(\theta), \forall \theta \in S^1\}$, $S^1 := \{y \in \mathbb{R}^2 : |y| = 1\}$, $\mathcal{M}$ is a C^2 connected sub-manifold of a Hilbert space H, $\mathcal{M} = -\mathcal{M}$ and $J : H \to \mathbb{R}$ is a C^1 functional. For some differential equations, we want to show that any nodal solution which attains the above minimax level is foliated Schwarz symmetric with respect to some point $P \in S^{N-1}$.

Let $\Omega \subseteq \mathbb{R}^N$ be a symmetric domain, a continuous function $u(x) : \Omega \to \mathbb{R}$ is called to be foliated Schwarz symmetric with respect to some point $P \in S^{N-1} = \{y \in \mathbb{R}^N : |y| = 1\}$ if u depends on $r = |x| > 0$ and $\theta = \arccos(\frac{x}{|x|} \cdot P)$ and u is nonincreasing in θ for any fixed $r > 0$.

We focus on the topic of the eigenvalue problem for the scalar field equation

$$-\Delta u + V(x)u = \lambda |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 3$, $p \in (2, 2^*]$ $(2^* = \frac{2N}{N-2})$, $V \in L^\infty(\mathbb{R}^N)$.

For $u \in H^1(\mathbb{R}^N)$, let

$$J(u) = \int_{\mathbb{R}^N} [|\nabla u|^2 + V(x)u^2] \mathrm{d}x, \quad I(u) = \int_{\mathbb{R}^N} |u|^p \mathrm{d}x.$$

Thus, we define

$$\mathcal{M} = \{u \in H^1(\mathbb{R}^N) : I(u) = 1\}, \quad \lambda_2 := \inf_{\gamma \in \Gamma_2} \max_{u \in \gamma(S^1)} J(u),$$

where $\Gamma_2 := \{\gamma \in C(S^1, \mathcal{M}) : \gamma(-\theta) = -\gamma(\theta), \forall \theta \in S^1\}$.

We use the tool of polarization (see [2,3,8,9,14]) to prove that any sign changing solution of (1.1) on $\mathcal{M}$ corresponding to $\lambda = \lambda_2$ is foliated Schwarz symmetric.

Theorem 1.1 *Suppose that $p \in (2, 2^*]$, $\lambda_2 > 0$ and that $V \in L^\infty(\mathbb{R}^N)$ is radially symmetric. If u is a nodal solution of (1.1) on $\mathcal{M}$ corresponding to $\lambda = \lambda_2$, then u is foliated Schwarz symmetric with respect to some point $P \in S^{N-1}$, where $S^{N-1} = \{y \in \mathbb{R}^N : |y| = 1\}$.*

Remark 1.2 As $N = 2$, $2^* = +\infty$, then for $2 < p < +\infty$, Theorem 1.1 still holds.

Remark 1.3 Let $V \in L^\infty(\mathbb{R}^N)$, $\lambda_2 > 0$ and u be a nodal solution of (1.1) on $\mathcal{M}$ corresponding to $\lambda = \lambda_2$. Assume that $\Omega = \mathbb{R}^N$ and β is the number defined for the equation (1.1) with $\lambda = 1$ in [2]. Then, by scaling methods, we have

$$\beta \leq (\frac{1}{2} - \frac{1}{p})\lambda_2^{\frac{p}{p-2}}.$$

Perera and Tintarev [9] proposed **an open problem** which states that if V is radially symmetric, whether every solution of (1.1) at level λ_2 is foliated Schwarz symmetry or not. Here, we answer this open problem when the solution is nodal. In [10], they obtain the existence of nonradial nodal solution of (1.1) under some conditions on $V(x)$. In this paper, we can show that this solution is foliated Schwarz symmetric with respect to some point $P \in S^{N-1}$.

Corollary 1.4 *Suppose that $p \in (2, 2^*)$, $V \in L^\infty(\mathbb{R}^N)$ is radially symmetric and satisfies*

$$V(x) \geq 0 \; \forall x \in \mathbb{R}^N, \quad \lim_{|x| \to \infty} V(x) = V^\infty > 0.$$

Suppose that $W(x) := V^\infty - V(x) \in L^{\frac{p}{p-2}}(\mathbb{R}^N)$ satisfies

$$W(x) \geq c_0 e^{-a|x|}, \forall x \in \mathbb{R}^N, \quad |W|_{\frac{p}{p-2}} < (2^{\frac{p}{p-2}} - 1)\lambda_1^\infty,$$

where c_0, a are some positive constants and

$$\lambda_1^\infty := \inf_{u\in\mathcal{M}} \int_{\mathbb{R}^N} [|\nabla u|^2 + V^\infty u^2] \mathrm{d}x.$$

Then, (1.1) *has a nodal solution on $\mathcal{M}$ corresponding to $\lambda = \lambda_2$, which is foliated Schwarz symmetric with respect to some point $P \in S^{N-1}$.*

For $u, v \in H^1(\mathbb{R}^N)$, we define the inner product as follows

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} uv \mathrm{d}x, \quad \forall u, v \in H^1(\mathbb{R}^N),$$

and the corresponding norm is denoted by $\|u\|$. For $1 \le p \le +\infty$ and $f \in L^p(\mathbb{R}^N)$, we denote by $|f|_p$ the usual L^p-norm of f, $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

This paper is organized as follows: in Sect. 2, we give some preliminaries; in Sect. 3, we prove Theorem 1.1.

2 Preliminaries

In this section, we give some preliminaries. Let $\mathcal{H}$ be all the closed affine halfspaces in $\mathbb{R}^N$ and $\mathcal{H}_0 = \{H : H \in \mathcal{H}, 0 \in \partial H\}$. For $H \in \mathcal{H}$, let $\sigma_H : \mathbb{R}^N \to \mathbb{R}^N$ be the reflection with respect to the boundary of H, and for a measurable function $u : \mathbb{R}^N \to \mathbb{R}$, let

$$u_H(x) = \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & x \in \mathbb{R}^N \backslash H \end{cases}$$

be the polarization of u with respect to H (see [2]).

Remark 2.1 From the definition of the polarization, it is easy to check that u_H is measurable.

The following definition can be found in [4] or [12].

Definition 2.2 Let $\Omega \subseteq \mathbb{R}^N$ be a symmetric domain, a continuous function $u(x) : \Omega \to \mathbb{R}$ is said to be foliated Schwarz symmetric with respect to some point $P \in S^{N-1} = \{y \in \mathbb{R}^N : |y| = 1\}$ if u depends on $r = |x| > 0$ and $\theta = \arccos(\frac{x}{|x|} \cdot P)$ and u is nonincreasing in θ for any fixed $r > 0$.

Remark 2.3 Let $\mathcal{H}_P := \{H : H \in \mathcal{H}_0$ and $P \in \mathring{H}\}$. The symmetrization A^P of a set $A \subseteq S_r^{N-1}(0) = \partial B_r(0)$ with respect to P is defined as the closed geodesic ball in $S_r^{N-1}(0)$ centered at rP which satisfies $H^{N-1}(A^P) = H^{N-1}(A)$, where H^{N-1} is the $N-1$-dimensional Hausdorff measure (see [1]). For a continuous function $u(x) : \Omega \to \mathbb{R}$, the foliated Schwarz symmetrization $u_P : \Omega \to \mathbb{R}$ of u with respect to P is defined by the condition

$$\{u_P \ge t\} \cap S_r^{N-1}(0) = [\{u \ge t\} \cap S_r^{N-1}(0)]^P, \quad \forall\, r > 0,\ t \in \mathbb{R}.$$

One can check that u_P is measurable and that u is foliated Schwarz symmetric with respect to P if and only if $u = u_P$ (See [14], p. 214).

The next two Lemmata can be found in [2], we omit their proofs.

Lemma 2.4 *For every measurable function* $u : \mathbb{R}^N \to \mathbb{R}$ *and for every* $H \in \mathcal{H}$*, there holds*

$$(u_H)^+ = (u^+)_H, (u_H)^- = (u^-)_H.$$

In the following, we may simply write $u_H^{\pm}$ due to Lemma 2.4.

Lemma 2.5 *If* $u \in W^{1,p}(\mathbb{R}^N)$ *and* $v \in L^p(\mathbb{R}^N)$ *for some* $1 \le p < +\infty$*, then for any* $H \in \mathcal{H}$*, we have*

$$u_H \in W^{1,p}(\mathbb{R}^N), \quad v_H \in L^p(\mathbb{R}^N),$$

and

$$\int_{\mathbb{R}^N} |\nabla u^{\pm}|^p \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u_H^{\pm}|^p \mathrm{d}x, \quad \int_{\mathbb{R}^N} |v^{\pm}|^p \mathrm{d}x = \int_{\mathbb{R}^N} |v_H^{\pm}|^p \mathrm{d}x.$$

Moreover, if $U : \mathbb{R}^N \to \mathbb{R}$ *is measurable and radially symmetric, and if*

$$\int_{\mathbb{R}^N} |U||v^{\pm}|^p \mathrm{d}x < +\infty,$$

then for any $H \in \mathcal{H}_0$*, we have*

$$\int_{\mathbb{R}^N} U|v^{\pm}|^p \mathrm{d}x = \int_{\mathbb{R}^N} U|v_H^{\pm}|^p \mathrm{d}x.$$

Lemma 2.6 *Suppose that* $P \in S^{N-1}$ *and* $u \in C(\mathbb{R}^N)$*. If* $u_H(x) = u(x)$ *for any* $H \in \mathcal{H}_P$*, then* u *is foliated Schwarz symmetric with respect to* P*.*

Proof According to Definition 2.2, we only need to prove that if $|x_1| = |x_2|$ and $\theta_1 = \arccos(\frac{x_1}{|x_1|} \cdot P) \le \theta_2 = \arccos(\frac{x_2}{|x_2|} \cdot P)$, then $u(x_1) \ge u(x_2)$.

If $\theta_1 < \theta_2$, then x_1 is closer to P than x_2; hence, there exists $H \in \mathcal{H}_P$ such that $x_1 \in H$ and $\sigma_H(x_1) = x_2$. Since $u_H(x_i) = u(x_i)$ for $i = 1, 2$, we have $u(x_1) \ge u \circ \sigma_H(x_1) = u(x_2)$ by the definition of u_H.

If $\theta_1 = \theta_2$, then there exists a sequence $\{\tilde{H}_n\}_{n=1}^{\infty} \subseteq \mathcal{H}_P$ with $x_1 \in \tilde{H}_n$ and $\sigma_{\tilde{H}_n}(x_1) \to x_2$ as $n \to \infty$. Since $u(x_1) \ge u \circ \sigma_{\tilde{H}_n}(x_1)$, we have $u(x_1) \ge \lim_{n\to\infty} u \circ \sigma_{\tilde{H}_n}(x_1) = u(x_2)$. □

3 Proof of Theorem 1.1

For $\lambda \in \mathbb{R}$, let $K_\lambda = \{u \in \mathcal{M} : J|'_{\mathcal{M}}(u) = 0, J(u) = \lambda\}$.

Lemma 3.1 *If* $\gamma_0 \in \Gamma_2$ *satisfying* $\max_{\theta \in S^1} J(\gamma_0(\theta)) = \lambda_2$*, then there exists* $\theta_0 \in S^1$ *such that* $J|'_{\mathcal{M}}(\gamma_0(\theta_0)) = 0$ *and* $J(\gamma_0(\theta_0)) = \lambda_2$*.*

Proof Let $\Sigma = \{\theta \in S^1 : J(\gamma_0(\theta)) = \lambda_2\}$. Then, Σ and $\gamma_0(\Sigma)$ are compact. If $J|'_{\mathcal{M}}(\gamma_0(\theta)) \ne 0$ for any $\theta \in \Sigma$, then there exist $\varepsilon, \delta > 0$ such that

$$\forall u \in J^{-1}([\lambda_2 - 2\varepsilon, \lambda_2 + 2\varepsilon]) \cap (\gamma_0(\Sigma))_{3\delta} : \|J|'_{\mathcal{M}}(u)\| \ge \frac{8\varepsilon}{\delta},$$

where $(\gamma_0(\Sigma))_{3\delta}$ is the 3δ neighbor of $\gamma_0(\Sigma)$ in $\mathcal{M}$. Therefore, by Deformation Lemma (see [11] or [15]), there exists $\eta : C([0, 1] \times \mathcal{M}, \mathcal{M})$ such that

(i) $\eta(1, J^{\lambda_2+\varepsilon}) \cap J((\gamma_0(\Sigma))_\delta) \subset J^{\lambda_2-\varepsilon}$,
(ii) $\eta(t, -u) = -\eta(t, u), \forall t \in [0, 1], u \in \mathcal{M}$,

(iii) $J(\eta(\cdot, u))$ is nonincreasing in $\mathbb{R}$ for any $u \in \mathcal{M}$.

Let $\tilde{\gamma_0}(\theta) = \eta(1, \gamma_0(\theta))$, then $\tilde{\gamma_0} \in \Gamma_2$ by (ii). Since $\gamma_0(S^1) \subseteq J^{\lambda_2}$, we have $\sup\limits_{\theta \in S^1} J(\tilde{\gamma_0}(S^1)) < \lambda_2$ by (i) and (iii), this contradicts to the definition of λ_2. □

For $u, v \in H^1(\mathbb{R}^N)\backslash\{0\}, v^{\pm} \neq 0$, define

$$\widehat{u} = \frac{u}{|u|_p} \text{ and } \gamma_v(\cos(t), \sin(t)) = \frac{\cos(t)\widehat{v^+} + \sin(t)\widehat{v^-}}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{1}{p}}},$$

then

$$\gamma_v \in \Gamma_2.$$

Now, we prove another Lemma which is similar to Proposition 4.2 of [2].

Lemma 3.2 *Suppose that* $\lambda_2 > 0$ *and* (1.1) *has a nodal solution on* $\mathcal{M}$ *corresponding to* $\lambda = \lambda_2$. *Then,*

$$\lambda_2 = \inf_{u \in \mathcal{M}, u^{\pm} \neq 0} \max_{t \in [0, 2\pi]} \frac{J(\widehat{u^+}) \cos^2(t) + J(\widehat{u^-}) \sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}}.$$

Moreover, if $u \in \mathcal{M}$ *satisfies* $u^{\pm} \neq 0$ *and*

$$\max_{t \in [0, 2\pi]} \frac{J(\widehat{u^+}) \cos^2(t) + J(\widehat{u^-}) \sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} = \lambda_2,$$

then there exits $t_0 \in [0, 2\pi]$ *such that*

$$J|'_{\mathcal{M}}(\gamma_u(\theta_0)) = 0 \text{ and } J(\gamma_u(\theta_0)) = \lambda_2.$$

Proof Let

$$\begin{aligned}\lambda_2' &= \inf_{u \in \mathcal{M}, u^{\pm} \neq 0} \max_{t \in (0, 2\pi)} \frac{J(\widehat{u^+}) \cos^2(t) + J(\widehat{u^-}) \sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} \\ &= \inf_{u \in \mathcal{M}, u^{\pm} \neq 0} \max_{S^1} J(\gamma_u).\end{aligned}$$

Hence,

$$\lambda_2' \geq \lambda_2.$$

On the other hand, let u_0 be a nodal solution of (1.1) on $\mathcal{M}$ corresponding to $\lambda = \lambda_2$, and then

$$-\Delta u_0 + V(x) u_0 = \lambda_2 |u_0|^{p-2} u_0.$$

Testing above equation with u_0^+ and u_0^- and integrating over $\mathbb{R}^N$ give

$$J(u_0^{\pm}) = \lambda_2 |u_0^{\pm}|_p^p, \quad J(\widehat{u_0^{\pm}}) = \frac{J(u_0^{\pm})}{|u_0^{\pm}|_p^2} = \lambda_2 |u_0^{\pm}|_p^{p-2}.$$

Therefore, according to the Hölder inequality and the positivity of λ_2, we have

$$\begin{aligned}\lambda_2' &\leq J(\gamma_{u_0}(\cos(t), \sin(t))) = \frac{J(\widehat{u_0^+}) \cos^2(t) + J(\widehat{u_0^-}) \sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} \\ &= \lambda_2 \frac{|u_0^+|_p^{p-2} \cos^2(t) + |u_0^-|_p^{p-2} \sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} \leq \lambda_2 (|u_0^+|_p^p + |u_0^-|_p^p)^{\frac{p-2}{p}} = \lambda_2.\end{aligned}$$

Hence,

$$\lambda_2 = \lambda_2'.$$

If

$$\max_{t\in(0,2\pi)} \frac{J(\widehat{u^+})\cos^2(t) + J(\widehat{u^-})\sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} = \max_{S^1} J(\gamma_u) = \lambda_2,$$

then Lemma 3.1 implies that there exits $t_0 \in [0, 2\pi]$ such that

$$J|'_{\mathcal{M}}(\gamma_u(\theta_0)) = 0 \text{ and } J(\gamma_u(\theta_0)) = \lambda_2.$$

□

Lemma 3.3 *If $\lambda_2 > 0$, $V \in L^\infty(\mathbb{R}^N)$ is symmetric and $u_0 \in K_{\lambda_2}$ is nodal, then u_{0H} is also nodal and u_{0H} belongs to K_{λ_2} for any $H \in \mathcal{H}_0$.*

Proof In the following, we fixed a $H \in \mathcal{H}_0$. Since u_0 satisfies (1.1) with $\lambda = \lambda_2$ and $V \in L^\infty(\mathbb{R}^N)$, we have $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0, 1)$ by elliptic regularity (see [6]). Hence,

$$u_{0H} = \frac{1}{2}u_0 + \frac{1}{2}u_0 \circ \sigma_H + \frac{1}{2}\chi_H|u_0 - u_0 \circ \sigma_H| - \frac{1}{2}(1 - \chi_H)|u_0 - u_0 \circ \sigma_H| \in C(\mathbb{R}^N).$$

Since u_0 is nodal, we have that u_{0H} is nodal and $\{y \in \mathbb{R}^N : |u_{0H}(y)| = 0\} \neq \emptyset$.

Note that

$$J(\gamma_{u_0}(\cos(t), \sin(t))) = \frac{J(\widehat{u_0^+})\cos^2(t) + J(\widehat{u_0^-})\sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}},$$

$$J(\gamma_{u_{0H}}(\cos(t), \sin(t))) = \frac{J(\widehat{u_{0H}^+})\cos^2(t) + J(\widehat{u_{0H}^-})\sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}}.$$

Lemmas 2.4 and 2.5 give

$$J(\widehat{u_0^+}) = J(\widehat{u_{0H}^+}),\ J(\widehat{u_0^-}) = J(\widehat{u_{0H}^-}),$$

hence

$$J(\gamma_{u_0}(\cos(t), \sin(t))) = J(\gamma_{u_{0H}}(\cos(t), \sin(t))),$$

$$\max_{t\in[0,2\pi]} J(\gamma_{u_0}(\cos(t), \sin(t))) = \max_{t\in[0,2\pi]} J(\gamma_{u_{0H}}(\cos(t), \sin(t))) = \lambda_2.$$

On the other hand,

$$\begin{aligned}
&J(\gamma_{u_{0H}}(\cos(t), \sin(t))) \\
&= J(\gamma_{u_0}(\cos(t), \sin(t))) \\
&= \lambda_2 \frac{|u_0^+|_p^{p-2}\cos^2(t) + |u_0^-|_p^{p-2}\sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} \\
&= \lambda_2 \frac{|u_{0H}^+|_p^{p-2}\cos^2(t) + |u_{0H}^-|_p^{p-2}\sin^2(t)}{(|\cos(t)|^p + |\sin(t)|^p)^{\frac{2}{p}}} \\
&\leq \lambda_2
\end{aligned}$$

with equality holds if and only if $|\tan(t)| = \frac{|u_0^+|_p}{|u_0^-|_p} = \frac{|u_{0H}^+|_p}{|u_{0H}^-|_p}$. Thus, Lemma 3.2 implies that there is a $t_0 \in [0, 2\pi]$ such that

$$|\tan(t_0)| = \frac{|u_0^+|_p}{|u_0^-|_p} = \frac{|u_{0H}^+|_p}{|u_{0H}^-|_p} \text{ and } \frac{\cos(t_0)\widehat{u_{0H}^+} + \sin(t_0)\widehat{u_{0H}^-}}{(|\cos(t_0)|^p + |\sin(t_0)|^p)^{\frac{1}{p}}} \in K_{\lambda_2},$$

hence

$$u_{0H} = u_{0H}^+ - u_{0H}^- \text{ or } |u_{0H}| = u_{0H}^+ + u_{0H}^- \in K_{\lambda_2}.$$

If $|u_{0H}| \in K_{\lambda_2}$, then

$$-\Delta|u_{0H}| + V(x)|u_{0H}| = \lambda_2|u_{0H}|^{p-1} \geq 0.$$

Since $V \in L^\infty(\mathbb{R}^N)$, elliptic regularities ([6]) give $|u_{0H}| \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0, 1)$. Maximal principle (see [5] or [7]) and the fact that $\{y \in \mathbb{R}^N : |u_{0H}(y)| = 0\} \neq \emptyset$ imply that $|u_{0H}| \equiv 0$, and this contradicts with the fact that $|u_{0H}|_p = |u|_p = 1$.

Therefore,

$$u_{0H} \in K_{\lambda_2}.$$

□

Now, we turn to the proof of Theorem 1.1.

Proof If u is nodal and solves (1.1) with $\lambda = \lambda_2$, then u_H also solves (1.1) with $\lambda = \lambda_2$ for any $H \in \mathcal{H}_0$ by Lemma 3.3. Since $V \in L^\infty(\mathbb{R}^N)$, elliptic regularities (see [6]) give $u, u_H \in C^{1,\alpha}(\mathbb{R}^N)$ for any $\alpha \in (0, 1)$. Choose $P \in S^{N-1}$ such that $u(P) = \max\{u(x) : x \in S^{N-1}\}$ and we will show that u is foliated Schwarz symmetric with respect to P.

Fix any $H \in \mathcal{H}_P$. In $\mathring{H} := H \backslash \partial H$, we have

$$|u - u \circ \sigma_H| = 2u_H - (u + u \circ \sigma_H),$$

therefore,

$$\begin{aligned}
&-\Delta|u - u \circ \sigma_H| + V(x)|u - u \circ \sigma_H| \\
&\quad = 2(-\Delta u_H + V(x)u_H) - (-\Delta u + V(x)u - \Delta u \circ \sigma_H + V(x)u \circ \sigma_H)) \\
&\quad = 2\lambda_2|u_H|^{p-2}u_H - (\lambda_2|u|^{p-2}u + \lambda_2|u \circ \sigma_H|^{p-2}u \circ \sigma_H) \\
&\quad = \lambda_2[(|u_H|^{p-2}u_H - |u|^{p-2}u) + (|u_H|^{p-2}u_H - |u \circ \sigma_H|^{p-2}u \circ \sigma_H))] \geq 0,
\end{aligned}$$

hence

$$|u - u \circ \sigma_H| > 0 \text{ in } \mathring{H} \text{ or } |u - u \circ \sigma_H| = 0 \text{ in } \mathring{H}.$$

Since $P \in \mathring{H}$, we have

$$u(P) \geq u \circ \sigma_H(P),$$

hence

$$u > u \circ \sigma_H \text{ in } \mathring{H} \text{ or } u = u \circ \sigma_H \text{ in } \mathring{H}.$$

Therefore,

$$u_H = u.$$

Thus, for any $H \in \mathcal{H}_P$, we have

$$u_H = u.$$

Lemma 2.6 implies u is foliated Schwarz symmetric with respect to P, and hence we get Theorem 1.1. □

Remark 3.4 The condition on $V(x)$ in Theorem 1.1 can be relaxed to "$V \in L^\infty_{loc}$". Replacing $H^1(\mathbb{R}^N)$ by

$$H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [|\nabla u|^2 + (1 + |V(x)|)u^2]\mathrm{d}x < \infty\},$$

the inner product on H can be defined as follows

$$(u, v) = \int_{\mathbb{R}^N} \nabla u \nabla v \mathrm{d}x + \int_{\mathbb{R}^N} (1 + |V(x)|)uv\mathrm{d}x, \quad \forall u, v \in H.$$

It is clear that H is a Hilbert space (see [13]).

Then, we define

$$\mathcal{M} = \{u \in H : I(u) = 1\}, \quad \lambda_2 := \inf_{\gamma \in \Gamma_2} \max_{u \in \gamma(S^1)} J(u),$$

where $\Gamma_2 := \left\{\gamma \in C(S^1, \mathcal{M}) : \gamma(-\theta) = -\gamma(\theta), \forall \theta \in S^1\right\}$.

Using the similar methods, we can prove Theorem 1.1 under the assumption "$V \in L^\infty_{loc}$".

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