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The Variable-Order Fractional Calculus of Variations

Springer

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Preface

This book intends to deepen the study of the fractional calculus, giving special emphasis to variable-order operators.

Fractional calculus is a recent field of mathematical analysis, and it is a generalization of integer differential calculus, involving derivatives and integrals of real or complex order [17, 27]. The first note about this idea of differentiation, for non-integer numbers, dates back to 1695, with a famous correspondence between Leibniz and L'Hôpital. In a letter, L'Hôpital asked Leibniz about the possibility of the order n in the notation $d^n y/dx^n$, for the nth derivative of the function y, to be a non-integer, $n = 1/2$. Since then, several mathematicians investigated this approach, like Lacroix, Fourier, Liouville, Riemann, Letnikov, Grünwald, Caputo, and contributed to the grown development of this field. Currently, this is one of the most intensively developing areas of mathematical analysis as a result of its numerous applications. The first book devoted to the fractional calculus was published by Oldham and Spanier in 1974, where the authors systematized the main ideas, methods, and applications about this field [18].

In the recent years, fractional calculus has attracted the attention of many mathematicians, but also some researchers in other areas like physics, chemistry, and engineering. As it is well known, several physical phenomena are often better described by fractional derivatives [13, 22, 30]. This is mainly due to the fact that fractional operators take into consideration the evolution of the system, by taking the global correlation, and not only local characteristics. Moreover, integer-order calculus sometimes contradict the experimental results, and therefore, derivatives of fractional order may be more suitable [14].

In 1993, Samko and Ross devoted themselves to investigate operators when the order α is not a constant during the process, but variable on time: $\alpha(t)$ [35]. An interesting recent generalization of the theory of fractional calculus is developed to allow the fractional order of the derivative to be non-constant, depending on time [9, 23, 24]. With this approach of variable-order fractional calculus, the non-local properties are more evident, and numerous applications have been found in physics, mechanics, control, and signal processing [10, 15, 25, 26, 28, 31, 42].

Although there are many definitions of fractional derivative, the most commonly used are the Riemann–Liouville, the Caputo, and the Grünwald–Letnikov derivatives. For more about the development of fractional calculus, we suggest [17, 18, 27, 34, 35].

One difficult issue that usually arises when dealing with such fractional operators is the extreme difficulty in solving analytically such problems [7, 44]. Thus, in most cases, we do not know the exact solution for the problem, and one needs to seek a numerical approximation. Several numerical methods can be found in the literature, typically applying some discretization over time or replacing the fractional operators by a proper decomposition [7, 44].

Recently, new approximation formulas were given for fractional constant-order operators, with the advantage that higher-order derivatives are not required to obtain a good accuracy of the method [6, 30, 31]. These decompositions only depend on integer-order derivatives, and by replacing the fractional operators that appear in the problem by them, one leaves the fractional context ending up in the presence of a standard problem, where numerous tools are available to solve them [2].

The first goal of this book is to extend such decompositions to Caputo fractional problems of variable order. For three types of Caputo derivatives with variable order, we obtain approximation formulas for the fractional operators and respective upper bounds for the errors.

Then, we focus our attention on a special operator introduced by Malinowska and Torres: the combined Caputo fractional derivative, which is an extension of the left and the right fractional Caputo derivatives [19]. Considering $\alpha, \beta \in (0, 1)$ and $\gamma \in [0, 1]$, the combined Caputo fractional derivative operator ${}^C D_\gamma^{\alpha,\beta}$ is a convex combination of the left and the right Caputo fractional derivatives, defined by

$$ {}^C D_\gamma^{\alpha,\beta} = \gamma\, {}_a^C D_t^\alpha + (1-\gamma)\, {}_t^C D_b^\beta. $$

We consider this fractional operator with variable fractional order, i.e., the combined Caputo fractional derivative of variable order:

$$ {}^C D_\gamma^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)} x(t) \; = \; \gamma_1\, {}_a^C D_t^{\alpha(\cdot,\cdot)} x(t) + \gamma_2\, {}_t^C D_b^{\beta(\cdot,\cdot)} x(t), $$

where $\gamma = (\gamma_1, \gamma_2) \in [0, 1]^2$, with γ_1 and γ_2 not both zero. With this fractional operator, we study different types of fractional calculus of variations problems, where the Lagrangian depends on the referred derivative.

The calculus of variations is a mathematical subject that appeared formally in the seventeenth century, with the solution to the Brachistochrone problem, that deals with the extremization (minimization or maximization) of functionals [43]. Usually, functionals are given by an integral that involves one or more functions or/and its derivatives. This branch of mathematics has proved to be relevant because of the numerous applications existing in real situations.

The fractional variational calculus is a recent mathematical field that consists in minimizing or maximizing functionals that depend on fractional operators (integrals or/and derivatives). This subject was introduced by Riewe in 1996, where the author generalizes the classical calculus of variations, by using fractional derivatives, and allows to obtain conservations laws with nonconservative forces such as friction [33, 34]. Later appeared several works on various aspects of the fractional calculus of variations and involving different fractional operators, like the Riemann–Liouville, the Caputo, the Gruńwald–Letnikov, the Weyl, the Marchaud or the Hadamard fractional derivatives [1, 3–5, 8, 11, 12, 16]. For the state of the art of the fractional calculus of variations, we refer the readers to the books [2, 20, 21].

Specifically, here we study some problems of the calculus of variations with integrands depending on the independent variable t, an arbitrary function x and a fractional derivative ${}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}x$. The endpoint of the cost integral, as well the terminal state, is considered to be free. The fractional problem of the calculus of variations consists in finding the maximizers or minimizers to the functional

$$\mathcal{J}(x,T) = \int_{a}^{T} L\left(t, x(t), {}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}x(t)\right)dt + \varphi(T, x(T)),$$

where ${}^{C}D_{\gamma}^{\alpha(\cdot,\cdot),\beta(\cdot,\cdot)}x(t)$ stands for the combined Caputo fractional derivative of variable fractional order, subject to the boundary condition $x(a) = x_a$. For all variational problems presented here, we establish necessary optimality conditions and transversality optimality conditions.

The book is organized in two parts, as follows. In the first part, we review the basic concepts of fractional calculus (Chap. 1) and of the fractional calculus of variations (Chap. 2). In Chap. 1, we start with a brief overview about fractional calculus and an introduction to the theory of some special functions in fractional calculus. Then, we recall several fractional operators (integrals and derivatives) definitions, and some properties of the considered fractional derivatives and integrals are introduced. In the end of this chapter, we review integration by parts formulas for different operators. Chapter 2 presents a short introduction to the classical calculus of variations and review different variational problems, like the isoperimetric problems or problems with variable endpoints. In the end of this chapter, we introduce the theory of the fractional calculus of variations and some fractional variational problems with variable order.

In the second part, we systematize some new recent results on variable-order fractional calculus of [37–41]. In Chap. 3, considering three types of fractional Caputo derivatives of variable order, we present new approximation formulas for those fractional derivatives and prove upper-bound formulas for the errors. In Chap. 4, we introduce the combined Caputo fractional derivative of variable-order and corresponding higher-order operators. Some properties are also given. Then, we prove fractional Euler–Lagrange equations for several types of fractional problems of the calculus of variations, with or without constraints.

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Any comments or suggestions related to the material here contained are more than welcome, and may be submitted by post or by electronic mail to the authors:

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